**Tasks**

# **Task 1. Ordinary differential equations.**

To solve the differential equation



with initial conditions



Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **variant** | ***t*0** | ***a*** | ***b*** | ***c*** |
| 1 | 0 | 4 | 2 | -1 |
| 2 | 1 | -4 | 1 | -2 |
| 3 | -1 | 1/4 | -1 | 2 |
| 4 | 1 | 9 | 3 | 0 |
| 5 | 2 | 1 | 2 | 2 |
| 6 | 0 | -1/4  | 0 | -1 |
| 7 | -1 | -9 | -1 | -2 |
| 8 | -2 | -1 | 1 | 4 |
| 9 | 0 | 1/9 | -1 | 0 |
| 10 | 1 | -1/9 | 1 | -1 |

**Task**.

1. Find the general solution of the given Cauchy problem. This is  if *a=λ*2 and  if *a=-λ*2.
2. Using the initial conditions, find the constant *c*1 and *c*2.
3. Put these constant to the formula of the general solution.
4. Make sure that the result satisfies, in reality the given equations and initial conditions.

# **Task 2. First order partial differential equations**

Consider the first order partial differential equation

 , 0<*x*<*L*, 0<*y*<*M*, (1)

where *a*, *L*, *M* are given constants. If *a*>0, then we can have the boundary conditions

  (2)

or

  (3)

If *a*<0, then we can have the boundary conditions

  (4)

or

 , (5)

where *ϕ* and *ψ* are given functions.

It is necessary to find the solution of the given problem, using characteristic method. Check that the result is, in reality, the solution of the problem.

Table of parameters

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **variant** | ***a*** | **boundary****conditions** | ***L*** | ***M*** | *ϕ*(*y*) | *ψ*(*x*) |
| 1 | 2 | (2) | 1 | 2 | sin π*y* | sin 2π*x* |
| 2 | -2 | (4) | 2 | 1 | cos 2π*y* | cos π*x* |
| 3 | 1/2 | (3) | π/2 | π | -sin *y* | -sin 2*x* |
| 4 | -1/2 | (5) | π | π/2 | -cos 2*y* | -cos *x* |
| 5 | 3 | (2) | π | π/2 | sin 2*x* | sin *y* |
| 6 | -3 | (4) | π/2 | π | cos *y* | cos 2*x* |
| 7 | 1/3 | (3) | 2 | 1 | -sin 2π*y* | -sin π*x* |
| 8 | -1/3 | (5) | 1 | 2 | -cos π*y* | -cos 2π*x* |

# **Task 3. Classification of partial differential equations and its canonic forms**

Determine the sets, where the given equation has the concrete type. Transform it to the canonic form for any considered type.

**Table of equations**

|  |  |
| --- | --- |
| variant | equation |
| 1 |  |
| 2 |  |
| 3 |  |
| 4 |  |
| 5 |  |
| 6 |  |
| 7 |  |
| 8 |  |

It is necessary perform the following steps:

1. Calculate the value of the discriminant *D.*
2. Using the sign of *D* determine the sets of the plane *xy*, where the equation has the concrete type.
3. For the hyperbolic case, write two characteristic equations.
4. Find its general solutions.
5. Write these general solutions in the form  and .
6. Determine the new variables  
7. Calculate the coefficients of the equation in the new variables by the given formulas.
8. Determine the canonic form of the given equation for the hyperbolic case.
9. For parabolic case, consider the unique characteristic equation, determine variable  by previous method with arbitrary variable *η*, and repeat the actions of hyperbolic case.
10. For elliptic case, consider the first characteristic equation with complex parameters, find its general solution, write it is the form , choose the functions *ξ* and *η* as new variables, and repeat the actions of hyperbolic case.

# **Task 4. Running waves**

Consider theCauchy problem for the vibrating string equation

*utt = uxx* , -∞ < *x* < ∞, *t* > 0;

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = 0.

The function *ϕ* is given, see the following images:



 Variant 1 Variant 2 Variant 3



 Variant 4 Variant 5 Variant 6



 Variant 7 Variant 8

It is necessary to use the D'Alembert formula and analyze the corresponding running waves. Show all steps of the phenomenon.

# **Task 5. Oscillation of the string with fixed ends**

Consider first order boundary problem for the vibrating string equation:

*utt = a2 uxx*, 0 < *x* < *L*, *t* > 0,

*u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0,

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*.

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | *L* | *a* | *ϕ*(*x*) | *ψ*(*x*) |
| 1 | 1 | 2 | - sin π*x* | 0 |
| 2 | π | 1 | 0 | sin *x* |
| 3 | 2 | ½ | sin 2π*x* | 0 |
| 4 | 2π | 2 | sin (*x/*2) | 0 |
| 5 | 1 | 1 | 0 | sin π*x* |
| 6 | π | ½ | sin *x* | 0 |
| 7 | 2 | 2 | 0 | sin 2π*x* |
| 8 | 2π | ½ | 0 | - sin (*x/*2) |

It is necessary to find the solution of the problem, to show the graph (position of the string for the different time points), and to give the physical interpretation of the results.

# **Task 6. Oscillation of the string with free ends**

Consider second order boundary problem for the vibrating string equation:

*utt = a2 uxx*, 0 < *x* < *L*, *t* > 0,

*ux*(0,*t*) = 0, *ux*(*L*,*t*) = 0, *t* > 0.

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*.

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | *L* | *a* | *ϕ*(*x*) | *ψ*(*x*) |
| 1 | π | 1 | 0 | sin *x* |
| 2 | 1 | 2 | - cos π*x* | 0 |
| 3 | 2π | 2 | cos (*x/*2) | 0 |
| 4 | 2 | ½ | cos 2π*x* | 0 |
| 5 | π | ½ | cos *x* | 0 |
| 6 | 1 | 1 | 0 | cos π*x* |
| 7 | 2π | ½ | 0 | - cos (*x/*2) |
| 8 | 2 | 2 | 0 | cos 2π*x* |

Task:

1. Find the solution of the problem.
2. Check that this is, in reality, the solution.
3. Show the graph (position of the string for the different time points).
4. Give the physical interpretation of the results.

# **Task 7. Oscillation of the string under exterior force**

Consider the movement of the under string exterior force characterized by the given function *f.* This is described by non-homogeneous vibrating string equation

*utt = a2 uxx* + *f*(*x*,*t*), 0 < *x* < *L*, *t* > 0.

Suppose the string is in the state of equilibrium and has zero velocity at the initial time. Then we have the initial conditions

*u*(*x*,0) = 0, *ut*(*x*,0) = 0, 0 < *x* < *L*.

The ends of the string can be free or fixed, i.e. we have one of the following boundary conditions

 *u*(0,*t*) = 0, *u*(*π*,*t*) = 0, *t* > 0; (\*)

 *ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0. (\*\*)

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | boundary condition | *L* | *a* | *f* |
| 1 | \* | 1 | 2 |  –sin π*x* |
| 2 | \*\* | π | 1 | cos *x* |
| 3 | \* | π | ½ | –2 sin *x* |
| 4 | \* | 1 | 3 | sin 2π*x* |
| 5 | \*\* | π | ½ | cos 2*x* |
| 6 | \*\* | 1 | 1 | –cos π*x* |
| 7 | \*\* | 1 | 2 | cos 2π*x* |
| 8 | \* | π | ½ | sin 2*x* |

Task:

1. Determine the solution of the problem as sinus Fourier series for the boundary conditions (\*) and cosine Fourier series for the boundary conditions (\*\*).
2. Find the Fourier coefficient of the parameters of the system.
3. Solves ordinary differential equations with respect to the Fourier coefficient of the solution of the problem.
4. Check that this is, in reality the solution of the boundary problem.
5. Show the graph (position of the string for the different time points).
6. Give the physical interpretation of the results.

# **Task 8. Heat transfer with knows temperature at the ends with fixed ends**

Consider first order boundary problem for the heat equation:

*ut = a2 uxx*, 0 < *x* < *L*, *t* > 0,

*u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0,

*u*(*x*,0) = *ϕ*(*x*), 0 < *x* < *L*.

Table of parameters

|  |  |  |  |
| --- | --- | --- | --- |
| Variant | *L* | *a* | *ϕ*(*x*) |
| 1 | π | 1 | - sin 2x |
| 2 | 1 | 2 | - sin π*x* |
| 3 | 2π | 2 | sin (*x/*2) |
| 4 | 2 | ½ | sin 2π*x* |
| 5 | π | ½ | sin *x* |
| 6 | 1 | 1 | - sin 2π*x* |
| 7 | 2π | ½ | sin 2π*x* |
| 8 | 2 | 2 | - sin 2*x* |

Task:

1. Find the solution of the problem.
2. Check that this is, in reality, the solution.
3. Show the graph (temperature of the body for the different time).
4. Give the physical interpretation of the results.

### Task. **Variational method for the minimization problem**

Consider the problem of minimization of the functional



on the class of functions that equal to zero on the boundary of the given rectangle

Table of parameters

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **variant** | ***a*** | ***b*** | ***c*** | ***d*** | ***F*** |
| 1 | 0 | 0 | 1 | 1 |  |
| 2 | -1 | -1 | 0 | 0 |  |
| 3 | 0 | -1 | 1 | 0 |  |
| 4 | -1 | 0 | 0 | 1 |  |
| 5 | 0 | -1 | 2 | 0 |  |
| 6 | -2 | 0 | 0 | 1 |  |
| 7 | -1 | 0 | 1 | 1 |  |
| 8 | 0 | -1 | 1 | 1 |  |

Task: It is necessary to transform the minimization problem to the partial differential equation, using the variational method.

# **Task 9. Heat transfer for the boundary isolated body.**

Consider second order boundary problem for the heat equation:

*ut = a2 uxx*, 0 < *x* < *L*, *t* > 0,

*ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0.

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*.

Table of parameters

|  |  |  |  |
| --- | --- | --- | --- |
| variant | *L* | *a* | *ϕ*(*x*) |
| 1 | π | ½ | -cos *x* |
| 2 | 2π | 1 | cos *x* |
| 3 | 1 | 2 | cos 2π*x* |
| 4 | 2 | ½ | -cos π*x* |
| 5 | π/2 | 2 | cos 2*x* |
| 6 | π | 2 | -cos 2*x* |
| 7 | 1 | 3 | -cos 3π*x* |
| 8 | 2π | 2 |  cos *x* |

Task:

1. Find the solution of the problem.
2. Check that this is, in reality, the solution.
3. Show the graph (temperature of the body for the different time).
4. Give the physical interpretation of the results.

# **Task 10. Heat transfer under the heat source.**

Consider the heat transfer under the exterior heat source characterized by the given function *f.* This phenomenon is described by non-homogeneous heat equation

*ut = a2 uxx* + *f*(*x*,*t*), 0 < *x* < *L*, *t* > 0.

Suppose the initial temperature is zero. Then we have the initial condition

*u*(*x*,0) = 0, 0 < *x* < *L*.

The temperature or the heat flux are zero at the ends, i.e. we have one of the following boundary conditions

 *u*(0,*t*) = 0, *u*(*π*,*t*) = 0, *t* > 0; (\*)

 *ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0. (\*\*)

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | boundary condition | *L* | *a* | *f* |
| 1 | \*\* | π | 1 | cos *x* |
| 2 | \* | 1 | 2 |  –sin π*x* |
| 3 | \* | 1 | 3 | sin 2π*x* |
| 4 | \* | π | ½ | –2 sin *x* |
| 5 | \*\* | 1 | 1 | –cos π*x* |
| 6 | \*\* | π | ½ | cos 2*x* |
| 7 | \* | π | ½ | sin 2*x* |
| 8 | \*\* | 1 | 2 | cos 2π*x* |

Task:

1. Determine the solution of the problem as sinus Fourier series for the boundary conditions (\*) and cosine Fourier series for the boundary conditions (\*\*).
2. Find the Fourier coefficient of the parameters of the system.
3. Solves ordinary differential equations with respect to the Fourier coefficients of the solution of the problem.
4. Check that this is, in reality the solution of the boundary problem.
5. Show the graph (temperature distribution for the different time points).
6. Give the physical interpretation of the results.

# **Task 11. Variational method for the minimization problem**

Consider the problem of minimization of the functional



on the class of functions that equal to zero on the boundary of the given rectangle

Table of parameters

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **variant** | ***a*** | ***b*** | ***c*** | ***d*** | ***F*** |
| 1 | 0 | 0 | 1 | 1 |  |
| 2 | -1 | -1 | 0 | 0 |  |
| 3 | 0 | -1 | 1 | 0 |  |
| 4 | -1 | 0 | 0 | 1 |  |
| 5 | 0 | -1 | 2 | 0 |  |
| 6 | -2 | 0 | 0 | 1 |  |
| 7 | -1 | 0 | 1 | 1 |  |
| 8 | 0 | -1 | 1 | 1 |  |

Task: It is necessary to transform the minimization problem to the partial differential equation, using the variational method.

# **Task 12. Field potential of the charge circle**

Consider the electrostatic field of the charged circle of the radium *a*. This phenomenon is described by the Laplace equation in the polar coordinates



with boundary condition

*u*(*a*,*ϕ*) = *f*(*ϕ*).

The boundary problem can be interior or exterior.

Table of parameters

|  |  |  |  |
| --- | --- | --- | --- |
| **variant** | **problem** | ***a*** | ***f* (*ϕ*)** |
| 1 | interior | 2 | - sin *x* |
| 2 | exterior | 3 | cos *x* |
| 3 | interior | ½ | sin 2*x* |
| 4 | exterior | 2 | -sin *x* |
| 5 | interior | 3 | -cos *x* |
| 6 | exterior | ½ | -sin 2*x* |
| 7 | interior | 2 | cos 2*x* |
| 8 | exterior | ½ | -cos 2*x* |

Task:

1. Find the solution of the problem.
2. Check that this is, in reality the solution.
3. Show the graph (position of the string for the different time points).
4. Give the physical interpretation of the results.

# **Task 13. Finite difference method**

Consider the extended heat equation with initial condition

*u*(*x*,0) = *ϕ*(*x*), 0<*x<L*

and different variant of the boundary conditions.

Table of parameter

|  |  |  |  |
| --- | --- | --- | --- |
| **variant** | **equation** | **left boundary condition** | **right boundary condition** |
| **1** | *ut* = *a*2*uxx +ux* + *f*(*x*,*t*) | *u*(0,*t*) = *p*(*t*) | *ux*(*L*,*t*) = -*q*(*t*) |
| **2** | *ut* = *a*2*uxx +u* + *f*(*x*,*t*) | *u*(0,*t*) = -*p*(*t*) | *ux*(*L*,*t*) = *q*(*t*) |
| **3** | *ut* = *a*2*uxx - ux* + *f*(*x*,*t*) | *u*(0,*t*) = *p*(*t*) | *u*(*L*,*t*) = -*q*(*t*) |
| **4** | *ut* = *a*2*uxx - u* + *f*(*x*,*t*) | *u*(0,*t*) = -*p*(*t*) | *u*(*L*,*t*) = *q*(*t*) |
| **5** | *ut* = *a*2*uxx - ux* + *f*(*x*,*t*) | *ux*(0,*t*) = *p*(*t*) | *u*(*L*,*t*) = -*q*(*t*) |
| **6** | *ut* = *a*2*uxx +u* + *f*(*x*,*t*) | *ux*(0,*t*) = -*p*(*t*) | *u*(*L*,*t*) = *q*(*t*) |
| **7** | *ut* = *a*2*uxx +ux* + *f*(*x*,*t*) | *ux*(0,*t*) = *p*(*t*) | *ux*(*L*,*t*) = -*q*(*t*) |
| **8** | *ut* = *a*2*uxx - u* + *f*(*x*,*t*) | *ux*(0,*t*) = -*p*(*t*) | *ux*(*L*,*t*) = *q*(*t*) |

Steps of the answer

1. Give the completely problem statement.
2. Give the approximation of the equation, initial condition and boundary conditions.
3. Write the completely algorithm of the finite difference method.